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On asymptotic properties of Freud–Sobolev orthogonal polynomials

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Abstract

In this paper we consider a Sobolev inner product

$$(f,g)_S = \int fg\, d\mu + \lambda \int f'g'\, d\mu \qquad (*)$$

and we characterize the measures μ for which there exists an algebraic relation between the polynomials, $\{P_n\}$, orthogonal with respect to the measure μ and the polynomials, $\{Q_n\}$, orthogonal with respect to $(*)$, such that the number of involved terms does not depend on the degree of the polynomials. Thus, we reach in a natural way the measures associated with a Freud weight. In particular, we study the case $d\mu = e^{-x^4}\, dx$ supported on the full real axis and we analyze the connection between the so-called Nevai polynomials (associated with the Freud

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weight e^{-x^4}) and the Sobolev orthogonal polynomials Q_n. Finally, we obtain some asymptotics for $\{Q_n\}$. More precisely, we give the relative asymptotics $\{Q_n(x)/P_n(x)\}$ on compact subsets of $\mathbb{C}\backslash\mathbb{R}$ as well as the outer Plancherel–Rotach-type asymptotics $\{Q_n(\sqrt[4]{n}x)/P_n(\sqrt[4]{n}x)\}$ on compact subsets of $\mathbb{C}\backslash[-a,a]$ being $a=\sqrt[4]{4/3}$.

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1. Introduction

The study of algebraic and analytic properties of polynomials orthogonal with respect to an inner product

$$(p,q)_S=\sum_{k=0}^{N}\int_{\mathbb{R}}\lambda_k p^{(k)}(x)q^{(k)}(x)\,d\mu_k(x), \tag{1}$$

where $(\mu_k)_{k=0}^N$ are measures supported on subsets of the real line and p, q are polynomials with real coefficients has attracted the interest of many researchers in the last years (see [10]). Despite the interest of this case (1), the approach was started for $N=1$. In such a situation several examples were very carefully analyzed from an algebraic point of view. The first one (see [7]) is related to Gegenbauer measures, i.e., $d\mu_0(x)=d\mu_1(x)=(1-x^2)^{\alpha}\chi_{[-1,1]}\,dx,\ \ \alpha>-1$ which represents a situation of measures with compact support. A second one, for unbounded support, is analyzed in [8] when $d\mu_0(x)=d\mu_1(x)=x^{\alpha}e^{-x}\chi_{\mathbb{R}^+}\,dx,\ \ \alpha>-1$. In both cases, the basic differences with the so-called standard case ($N=0$) are emphasized. In particular, the three-term recurrence relation for the orthogonal polynomials fails and, as a consequence, the study of algebraic and analytic properties of the corresponding sequences of orthogonal polynomials needs different tools.

In a more general framework, if μ_0 or μ_1 are classical measures (Jacobi, Laguerre, Hermite), then the basic idea is to consider a companion measure which satisfies the so-called coherence condition or symmetric coherence for symmetric measures (see [3]). The goal of coherence is the fact that we can establish a finite algebraic relation between the orthogonal polynomials, $\{P_n\}$, associated with μ_0 and the orthogonal polynomials, $\{Q_n\}$, with respect to the inner product (1) for $N=1$, the so-called Sobolev orthogonal polynomials. This algebraic relation plays an important role in the study of $\{Q_n\}$ since it allows to express the polynomials $\{Q_n\}$ in terms of the standard polynomials $\{P_n\}$ and thus it is possible to carry out a study of $\{Q_n\}$ from the algebraic, analytic and computational points of view. Notice that in [13] it was proved that if (μ_0,μ_1) is a coherent pair of measures, then one of them must be classical and its companion is a perturbation of it.

If both measures μ_0 and μ_1 have unbounded support then, except for coherent pairs, very few examples are known when μ_0 and μ_1 are non-classical measures with non-zero absolutely continuous part. However, in the bounded case, i.e., the measures have compact support, quite a few things are known when both measures are non-classical. For instance, a nice survey about asymptotics of Sobolev orthogonal polynomials is [10]. Indeed, one of the aims of our contribution is to analyze orthogonal polynomials for an inner product (1) when $d\mu_0(x) = d\mu_1(x) = e^{-x^4}\chi_{\mathbb{R}}\,dx$, an example of a non-classical measure. The sequence of standard polynomials orthogonal with respect to such kind of measures has been introduced by Nevai [14,15] in the framework of the so-called Freud measures. They belong to the set of semiclassical measures, i.e., the linear functional $u : \mathbb{P} \to \mathbb{R}$ given by $(u, p) = \int_{\mathbb{R}} p(x)\,d\mu(x)$, where $\mathbb{P}$ is the linear space of polynomials with real coefficients is such that there exist polynomials ϕ, ψ with $\deg \psi \geqslant 1$ and (see [9])

$$D(\phi u) = \psi u. \tag{2}$$

Indeed, Freud measures are defined by weight functions $w(x) = e^{-P}$ where P is a monic polynomial of degree $2n$.

For Sobolev inner products (1) with $N = 1$ and $\mu_0 = \mu_1 = \mu$ the following result is proved in [2].

Proposition 1. *If μ is a semiclassical measure such that* (2) *holds, then there exists a non-negative integer number s such that*

$$\phi(x)P_n(x) = \sum_{j=n-s}^{n+s'} \alpha_{n,j}Q_j(x), \quad \alpha_{n,\,n-s} \neq 0, \tag{3}$$

where $\deg \phi = s'$.

In what follows, we use the inner product

$$(p, q)_S = \int_{\mathbb{R}} pq\,d\mu + \lambda \int_{\mathbb{R}} p'q'\,d\mu, \quad \lambda > 0, \tag{4}$$

and we denote by $\{P_n\}$ the sequence of orthogonal polynomials associated with $\mu = \mu_0 = \mu_1$ and by $\{Q_n\}$ the sequence of orthogonal polynomials with respect to (4).

We are interested in a converse result of Proposition 1, i.e., if we consider an inner product (4), such that (3) holds, then the goal is to know what information about the measure μ can be given. Indeed, we get

Theorem 1. *Relation* (3) *holds if and only if the measure μ is semiclassical. Furthermore, the polynomials ϕ, ψ in* (2) *can explicitly be given.*

In particular, if $\phi \equiv 1$ then μ is a Freud measure.

Our second step is to analyze orthogonal polynomials associated with the Sobolev inner product (4) when $d\mu = e^{-x^4}\,dx$. In Section 3 we deduce the connection between the sequences $\{P_n\}$ and $\{Q_n\}$. In such a way we can obtain an explicit expression for $\{Q_n\}$ in terms of $\{P_n\}$. From it we deduce in Section 4 the relative asymptotics of Q_n

with respect to P_n as well as a Plancherel–Rotach-type asymptotics formula for Q_n. Here the scaling in the variable is needed.

2. Proof of Theorem 1

Let ϕ be a polynomial of degree s' such that

$$\phi(x)P_n(x) = \sum_{j=n-s}^{n+s'} \alpha_{n,j} Q_j(x), \quad n \geqslant s, \tag{5}$$

with $\alpha_{n,n-s} \neq 0$ and $s' \leqslant s$. From (5), we get

$$0 = (\phi(x)P_n(x), Q_j(x))_S, \quad j = 0, 1, \ldots, n-s-1, \tag{6}$$

$$0 \neq (\phi(x)P_n(x), Q_{n-s}(x))_S = \alpha_{n,n-s}(Q_{n-s}, Q_{n-s})_S. \tag{7}$$

From (6),

$$\begin{aligned} 0 &= \int_{\mathbb{R}} \phi(x)P_n(x)Q_j(x)\, d\mu + \lambda \int_{\mathbb{R}} (\phi(x)P_n(x))' Q_j'(x)\, d\mu \\ &= \int_{\mathbb{R}} P_n(x)[\phi(x)Q_j(x) + \lambda\phi'(x)Q_j'(x)]\, d\mu + \lambda \int_{\mathbb{R}} \phi(x)P_n'(x)Q_j'(x)\, d\mu. \end{aligned}$$

The degree of the polynomial inside the brackets is $s' + j$ and taking into account that $0 \leqslant j \leqslant n-s-1$, from the orthogonality of P_n with respect to μ we deduce

$$\int_{\mathbb{R}} \phi(x)P_n'(x)Q_j'(x)\, d\mu = 0, \quad \text{for } 0 \leqslant j \leqslant n-s-1.$$

This means that the polynomial $\phi P_n'$ is orthogonal to $\mathbb{P}_{n-s-2}$ with respect to the measure μ, i.e.,

$$\phi(x)P_n'(x) = \sum_{j=n-s-1}^{n-1+s'} b_{n,j} P_j(x). \tag{8}$$

On the other hand, from (7),

$$0 \neq \int_{\mathbb{R}} P_n(x)[\phi(x)Q_{n-s} + \lambda\, \phi'(x)Q_{n-s}'(x)]\, d\mu + \lambda \int_{\mathbb{R}} \phi(x)P_n'(x)Q_{n-s}'(x)\, d\mu.$$

The degree of the polynomial inside the brackets is $n - s + s' \leqslant n$.

If $s' < s$, then we get

$$\int_{\mathbb{R}} \phi(x)P_n'(x)Q_{n-s}'(x)\, d\mu \neq 0.$$

Taking into account (8) and the fact that

$$Q_{n-s}'(x) = (n-s)P_{n-s-1}(x) + \text{lower degree terms}, \tag{9}$$

we get

$$\int_{\mathbb{R}} \phi(x) P'_n(x) P_{n-s-1}(x)\, d\mu \neq 0,$$

i.e., in (8) $b_{n,n-s-1} \neq 0$.

Now, if $s' = s$,

$$0 \neq a \int_{\mathbb{R}} P_n^2(x)\, d\mu + \lambda \int_{\mathbb{R}} \phi(x) P'_n(x) Q'_{n-s}(x)\, d\mu,$$

where a is the leading coefficient of $\phi(x)$. Again, from (8) and (9), we get

$$0 \neq a \int_{\mathbb{R}} P_n^2(x)\, d\mu + \lambda(n-s) b_{n,n-s-1} \int_{\mathbb{R}} P_{n-s-1}^2(x)\, d\mu.$$

In other words, (7) becomes

$$\alpha_{n,n-s}(Q_{n-s}, Q_{n-s})_S = a \int_{\mathbb{R}} P_n^2(x)\, d\mu + \lambda(n-s) b_{n,n-s-1} \int_{\mathbb{R}} P_{n-s-1}^2(x)\, d\mu.$$

Thus, $b_{n,n-s-1} \neq 0$ if and only if

$$\alpha_{n,n-s}(Q_{n-s}, Q_{n-s})_S \neq a \int_{\mathbb{R}} P_n^2(x)\, d\mu.$$

Finally, we have $(u, \phi(x) P'_n(x)) = 0$, for $n \geqslant s+2$, and then we get $(\phi(x)u, P'_n(x)) = 0$, for $n \geqslant s+2$, i.e., $(D(\phi u), P_n(x)) = 0$, for $n \geqslant s+2$. Thus

$$D(\phi\, u) = \sum_{k=0}^{s+1} \beta_k \frac{P_k(x) u}{(u, P_k^2(x))},$$

where

$$\begin{aligned}\beta_k &= (D(\phi u), P_k(x)) = -(\phi u, P'_k(x)) = -(u, \phi P'_k(x)) \\ &= -\left(u, \sum_{j=0}^{s'+k-1} b_{k,j} P_j(x)\right) = -b_{k,0}.\end{aligned}$$

Then,

$$D(\phi u) = -\sum_{k=0}^{s+1} \frac{b_{k,0} P_k(x)}{(u, P_k^2(x))} u = \psi u,$$

where

$$\begin{aligned}\psi(x) &= -\sum_{k=0}^{s+1} \frac{b_{k,0} P_k(x)}{(u, P_k^2(x))} = -\sum_{k=0}^{s+1} \frac{(u, \phi(t) P'_k(t) P_k(x))}{(u, P_k^2(t))} \\ &= -(u, \phi(t) K_{s+1}^{(1,0)}(t,x)).\end{aligned}$$

Here,

$$K_n(t,x) = \sum_{j=0}^{n} \frac{P_k(t) P_k(x)}{(u, P_k^2(x))}$$

is the nth kernel polynomial associated with the sequence (P_n) and we denote $K_n^{(1,0)}(t,x) = \frac{\partial}{\partial t} K_n(t,x)$.

Notice that $\deg(\phi) = s'$ and $1 \leqslant \deg(\psi) \leqslant s+1$. According to the definition of the order of a semiclassical linear functional (see [9]), the order of u is, at most, $\max\{s'-2, s\}$. $\square$

The simplest case corresponds to $\phi(x) = 1$. In this situation

$$Du = -(u, K_{s+1}^{(1,0)}(t,x))u, \quad \text{with } s \geqslant 0.$$

If u is induced by an absolutely continuous measure μ, i.e., $d\mu(x) = w(x)\,dx$, then $w'(x) = -\psi(x)w(x)$ and $w(x) = \exp\left(-\int \psi(x)\,dx\right)$. Thus, we obtain a Freud weight.

3. The Freud weight e^{-x^4} and the Sobolev orthogonal polynomials

Let $\{P_n\}$ be the sequence of monic polynomials orthogonal with respect to the weight function $d\mu = e^{-x^4}\,dx$ supported on $\mathbb{R}$. As we mentioned in Section 1, they have been considered by Nevai [14,15]. These polynomials satisfy a three-term recurrence relation

$$xP_n(x) = P_{n+1}(x) + c_n P_{n-1}(x), \quad n \geqslant 1,$$

with initial conditions $P_0(x) = 1$ and $P_1(x) = x$, where the parameters c_n satisfy a non-linear recurrence relation (see [4])

$$n = 4c_n(c_{n+1} + c_n + c_{n-1}), \quad n \geqslant 1, \tag{10}$$

with $c_0 = 0$ and $c_1 = \Gamma(3/4)/\Gamma(1/4)$.

On the other hand, from (8) with $s' = 0$ ($\phi \equiv 1$) and $\int \psi(x)\,dx = x^4$, i.e., $\psi(x) = 4x^3$ ($s = 2$), the polynomials $\{P_n\}$ satisfy a structure relation

$$P_n'(x) = nP_{n-1}(x) + d_n P_{n-3}(x), \quad n \geqslant 3,$$

where

$$\begin{aligned} d_n &= \frac{\int_{-\infty}^{\infty} P_n'(x)P_{n-3}(x)e^{-x^4}\,dx}{\int_{-\infty}^{\infty} P_{n-3}^2(x)e^{-x^4}\,dx} = \frac{-\int_{-\infty}^{\infty} P_n(x)[P_{n-3}'(x) - 4x^3P_{n-3}(x)]e^{-x^4}\,dx}{\int_{-\infty}^{\infty} P_{n-3}^2(x)e^{-x^4}\,dx} \\ &= \frac{4\int_{-\infty}^{\infty} P_n^2(x)e^{-x^4}\,dx}{\int_{-\infty}^{\infty} P_{n-3}^2(x)e^{-x^4}\,dx} = 4c_nc_{n-1}c_{n-2}, \quad n \geqslant 3. \end{aligned}$$

We consider the Sobolev inner product

$$(p,q)_S = \int_{-\infty}^{\infty} p(x)q(x)e^{-x^4}\,dx + \lambda \int_{-\infty}^{\infty} p'(x)q'(x)e^{-x^4}\,dx, \quad p,q \in \mathbb{P},$$

and let $\{Q_n\}$ be the corresponding sequence of monic orthogonal polynomials. Taking into account Proposition 1 as well as the fact that $Q_n(-x) = (-1)^n Q_n(x)$ we get

Proposition 2. *The polynomial* $\{P_n\}$ *and* $\{Q_n\}$ *are related by*

$$P_n(x) = Q_n(x) + \lambda_{n-2} Q_{n-2}(x), \quad n \geqslant 3. \tag{11}$$

Proof. From

$$P_n(x) = Q_n(x) + \sum_{j=0}^{n-2} \lambda_{n,j} Q_j(x),$$

for $0 \leqslant j \leqslant n-2$, we get

$$\begin{aligned}\lambda_{n,j} &= \frac{(P_n(x), Q_j(x))_S}{||Q_j||_S^2} = \frac{\lambda \int_{-\infty}^{\infty} P_n'(x) Q_j'(x) e^{-x^4}\, dx}{||Q_j||_S^2} \\ &= \frac{\lambda \int_{-\infty}^{\infty} 4 P_n(x) x^3 Q_j'(x) e^{-x^4}\, dx}{||Q_j||_S^2}.\end{aligned}$$

This expression vanishes for $j < n-2$. For $j = n-2$ we get

$$\lambda_{n,n-2} := \lambda_{n-2} = 4\lambda(n-2) \frac{\int_{-\infty}^{\infty} P_n^2(x) e^{-x^4}\, dx}{||Q_{n-2}||_S^2} > 0. \qquad \square \tag{12}$$

On the other hand, we can observe that $Q_i(x) = P_i(x)$, $i = 0, 1, 2$.

Notice that

$$||P_n||_S^2 = ||Q_n||_S^2 + \lambda_{n-2}^2 ||Q_{n-2}||_S^2,$$

with

$$\begin{aligned}||P_n||_S^2 &= \int_{-\infty}^{\infty} P_n^2(x) e^{-x^4}\, dx + \lambda \int_{-\infty}^{\infty} (P_n'(x))^2 e^{-x^4}\, dx = \int_{-\infty}^{\infty} P_n^2(x) e^{-x^4}\, dx \\ &\quad + \lambda \left[n^2 \int_{-\infty}^{\infty} P_{n-1}^2(x) e^{-x^4}\, dx + d_n^2 \int_{-\infty}^{\infty} P_{n-3}^2(x) e^{-x^4}\, dx \right],\end{aligned} \tag{13}$$

and using (12) we have

$$\begin{aligned}&||Q_n||_S^2 + \lambda_{n-2}^2\, ||Q_{n-2}||_S^2 \\ &\quad = 4\lambda \left(\frac{n \int_{-\infty}^{\infty} P_{n+2}^2(x) e^{-x^4}\, dx}{\lambda_n} + (n-2)\, \lambda_{n-2} \int_{-\infty}^{\infty} P_n^2(x) e^{-x^4}\, dx \right).\end{aligned} \tag{14}$$

Gathering (13) and (14) we obtain

$$\begin{aligned}&\int_{-\infty}^{\infty} P_n^2(x) e^{-x^4}\, dx + \lambda \left(n^2 \int_{-\infty}^{\infty} P_{n-1}^2(x) e^{-x^4}\, dx + d_n^2 \int_{-\infty}^{\infty} P_{n-3}^2(x) e^{-x^4}\, dx \right) \\ &\quad = 4\lambda \left(\frac{n}{\lambda_n} \int_{-\infty}^{\infty} P_{n+2}^2(x) e^{-x^4}\, dx + (n-2)\lambda_{n-2} \int_{-\infty}^{\infty} P_n^2(x) e^{-x^4}\, dx \right).\end{aligned}$$

Then, dividing by $\int_{-\infty}^{\infty} P_n^2(x)e^{-x^4}\,dx$ we get

$$1+\lambda\left(\frac{n^2}{c_n}+\frac{d_n^2}{c_nc_{n-1}c_{n-2}}\right)=4\lambda\left(\frac{n}{\lambda_n}c_{n+2}c_{n+1}+(n-2)\lambda_{n-2}\right),$$

or, equivalently,

$$1+\lambda\left(\frac{n^2}{c_n}+16c_nc_{n-1}c_{n-2}\right)=4\lambda\left(\frac{n}{\lambda_n}c_{n+2}c_{n+1}+(n-2)\lambda_{n-2}\right),\quad n\geqslant 3.$$

Finally,

$$\frac{1}{\lambda}+\frac{n^2}{c_n}+16c_nc_{n-1}c_{n-2}=4\left(\frac{n}{\lambda_n}c_{n+2}c_{n+1}+(n-2)\lambda_{n-2}\right),\quad n\geqslant 3, \tag{15}$$

with initial conditions

$$\lambda_1=\frac{4c_3c_2c_1}{1+c_1\lambda^{-1}},$$
$$\lambda_2=\frac{8c_4c_3c_2}{4+c_2\lambda^{-1}}.$$

Notice that for $n=2m$, an even non-negative integer number, we can assume

$$\lambda_{2m}=\frac{q_{m-1}(\lambda^{-1})}{q_m(\lambda^{-1})},\quad m\geqslant 1,$$

where q_m is a polynomial of degree m. Thus, for $m\geqslant 1$, (15) becomes

$$c_{2m+2}c_{2m+1}\frac{8mq_m(\lambda^{-1})}{q_{m-1}(\lambda^{-1})}+4(2m-2)\frac{q_{m-2}(\lambda^{-1})}{q_{m-1}(\lambda^{-1})}=\lambda^{-1}+\frac{4m^2}{c_{2m}}+16c_{2m}c_{2m-1}c_{2m-2},$$

or, equivalently,

$$q_m(\lambda^{-1})=\frac{q_{m-1}(\lambda^{-1})}{8mc_{2m+2}c_{2m+1}}\left(\lambda^{-1}+\frac{4m^2}{c_{2m}}+16c_{2m}c_{2m-1}c_{2m-2}\right)-\left(\frac{m-1}{m}\right)\frac{q_{m-2}(\lambda^{-1})}{c_{2m+2}c_{2m+1}}.$$

If $\tilde{q}_m$ denotes the monic polynomial associated with q_m, i.e., $q_m=s_m\tilde{q}_m$, we get

$$\tilde{q}_m(\lambda^{-1})=\left(\lambda^{-1}+\frac{4m^2}{c_{2m}}+16c_{2m}c_{2m-1}c_{2m-2}\right)\tilde{q}_{m-1}(\lambda^{-1})-64(m-1)^2c_{2m}c_{2m-1}\tilde{q}_{m-2}(\lambda^{-1}),\quad m\geqslant 2,$$

with initial conditions

$$\tilde{q}_0(\lambda^{-1})=1\quad\text{and}\quad\tilde{q}_1(\lambda^{-1})=\lambda^{-1}+\frac{4}{c_2}.$$

On the other hand, for $n = 2m-1$, an odd nonnegative integer number, we can assume

$$\lambda_{2m-1} = \frac{r_{m-1}(\lambda^{-1})}{r_m(\lambda^{-1})}, \quad m \geqslant 1,$$

where r_m is a polynomial of degree m. Thus, for $m \geqslant 2$, (15) becomes

$$c_{2m+1}c_{2m}\frac{4(2m-1)r_m(\lambda^{-1})}{r_{m-1}(\lambda^{-1})} + 4(2m-3)\frac{r_{m-2}(\lambda^{-1})}{r_{m-1}(\lambda^{-1})} = \lambda^{-1} + \frac{(2m-1)^2}{c_{2m-1}} + 16c_{2m-1}c_{2m-2}c_{2m-3},$$

or, equivalently,

$$r_m(\lambda^{-1}) = \frac{r_{m-1}(\lambda^{-1})}{4(2m-1)c_{2m+1}c_{2m}}\left(\lambda^{-1} + \frac{(2m-1)^2}{c_{2m-1}} + 16c_{2m-1}c_{2m-2}c_{2m-3}\right) - \frac{2m-3}{(2m-1)c_{2m+1}c_{2m}}r_{m-2}(\lambda^{-1}).$$

If $r_m = t_m\tilde{r}_m$ where $\tilde{r}_m$ denotes the monic polynomial associated with r_m, then we get

$$\tilde{r}_m(\lambda^{-1}) = \left(\lambda^{-1} + \frac{(2m-1)^2}{c_{2m-1}} + 16c_{2m-1}c_{2m-2}c_{2m-3}\right)\tilde{r}_{m-1}(\lambda^{-1}) - 16(2m-3)^2c_{2m-1}c_{2m-2}\tilde{r}_{m-2}(\lambda^{-1}), \quad m \geqslant 2,$$

with initial conditions

$$\tilde{r}_0(\lambda^{-1}) = 1 \quad \text{and} \quad \tilde{r}_1(\lambda^{-1}) = \lambda^{-1} + \frac{1}{c_1}.$$

As a conclusion, $\{\tilde{q}_m\}$ and $\{\tilde{r}_m\}$ are sequences of monic orthogonal polynomials.

4. Asymptotics of Q_n

First, we establish the asymptotic behavior of the sequence $\{\lambda_n\}$ which appears in (11).

Proposition 3. *For the sequence $\{\lambda_n/\sqrt{n}\}$ we get the upper bound*

$$\frac{\lambda_n}{\sqrt{n}} < \frac{\sqrt{5}}{3}, \quad n \geqslant 1.$$

Furthermore, the sequence $\{\lambda_n/\sqrt{n}\}$ is convergent and

$$\lim_{n\to\infty}\frac{\lambda_n}{\sqrt{n}} = \frac{1}{6\sqrt{3}}. \tag{16}$$

Proof. By the extremal property of $||Q_n||_S^2$ we have

$$||Q_n||_S^2 \geqslant \int_{-\infty}^{\infty} P_n^2(x)e^{-x^4}\,dx + \lambda n^2 \int_{-\infty}^{\infty} P_{n-1}^2(x)e^{-x^4}\,dx.$$

Thus, for $n \geqslant 1$, from (11)

$$\begin{aligned}\frac{\lambda_n}{\sqrt{n}} &= 4\lambda\sqrt{n}\,\frac{\int_{-\infty}^{\infty} P_{n+2}^2(x)e^{-x^4}\,dx}{||Q_n||_S^2}\\ &\leqslant 4\lambda\sqrt{n}\,\frac{\int_{-\infty}^{\infty} P_{n+2}^2(x)e^{-x^4}\,dx}{\int_{-\infty}^{\infty} P_n^2(x)e^{-x^4}\,dx + \lambda n^2 \int_{-\infty}^{\infty} P_{n-1}^2(x)e^{-x^4}\,dx}\\ &= 4\lambda\sqrt{n}\,\frac{c_{n+2}c_{n+1}c_n}{c_n + \lambda n^2}.\end{aligned} \tag{17}$$

From the recurrence relation (10) we have, for $n \geqslant 2$,

$$4c_n^2 = n - 4c_n(c_{n+1} + c_{n-1}) < n \Rightarrow c_n < \frac{\sqrt{n}}{2},$$

but simple computations show that the above inequality also holds for $n = 1$, that is,

$$c_n < \frac{\sqrt{n}}{2}, \quad n \geqslant 1. \tag{18}$$

Now, using this inequality in (10) we obtain

$$\frac{n}{4} = c_n(c_{n+1} + c_n + c_{n-1}) < \frac{3}{2}\sqrt{n+1}\,c_n$$

and, then $c_n > \frac{n\sqrt{n+1}}{6(n+1)}$, for $n \geqslant 2$, but again simple computations prove that this inequality is true for $n = 1$, and, therefore,

$$c_n > \frac{n}{6\sqrt{n+1}}, \quad n \geqslant 1. \tag{19}$$

We use relations (18) and (19) in (17) obtaining, for $n \geqslant 2$,

$$\frac{\lambda_n}{\sqrt{n}} < \frac{\lambda\sqrt{(n+2)(n+1)}}{\frac{1}{3\sqrt{n+1}} + 2\lambda n} < \frac{\sqrt{(n+2)(n+1)}}{2n} \leqslant \frac{\sqrt{5}}{3}, \quad n \geqslant 3,$$

and straightforward computations in (12) show that this inequality holds for $n = 1, 2$. Thus,

$$\frac{\lambda_n}{\sqrt{n}} < \frac{\sqrt{5}}{3} < 1, \quad n \geqslant 1. \tag{20}$$

On the other hand, relation (15) can be rewritten as

$$\lambda_n = \frac{4nc_{n+2}c_{n+1}}{\frac{1}{\lambda} + \frac{n^2}{c_n} + 16c_nc_{n-1}c_{n-2} - 4(n-2)\lambda_{n-2}}, \quad n \geqslant 3$$

and from here we get for $n \geqslant 3$,

$$\frac{\lambda_n}{\sqrt{n}} = \frac{1}{\frac{1}{4\lambda\sqrt{n}c_{n+2}c_{n+1}} + \frac{1}{4}\left(\frac{n^{3/2}}{c_n c_{n+1} c_{n+2}} + \frac{16 c_n c_{n-1} c_{n-2}}{\sqrt{n} c_{n+1} c_{n+2}}\right) - \frac{n-2}{c_{n+1}c_{n+2}}\sqrt{\frac{n-2}{n}}\frac{\lambda_{n-2}}{\sqrt{n-2}}}. \tag{21}$$

Denoting

$$B(n) = \frac{1}{4\lambda\sqrt{n}c_{n+2}c_{n+1}} + \frac{1}{4}\left(\frac{n^{3/2}}{c_n c_{n+1} c_{n+2}} + \frac{16 c_n c_{n-1} c_{n-2}}{\sqrt{n} c_{n+1} c_{n+2}}\right), \tag{22}$$

$$C(n) = \frac{n-2}{c_{n+1}c_{n+2}}\sqrt{\frac{n-2}{n}}, \tag{23}$$

then (21) becomes

$$\frac{\lambda_n}{\sqrt{n}} = \frac{1}{B(n) - C(n)\frac{\lambda_{n-2}}{\sqrt{n-2}}}. \tag{24}$$

Taking into account that in [4] an asymptotic expansion of c_n was established, in particular

$$\lim_{n\to\infty} \frac{c_n}{\sqrt{n}} = \frac{1}{2\sqrt{3}}, \tag{25}$$

using (25) we get

$$\lim_{n\to\infty} B(n) = \frac{20\sqrt{3}}{3}, \quad \lim_{n\to\infty} C(n) = 12. \tag{26}$$

Therefore, if the sequence $\{\lambda_n/\sqrt{n}\}$ converges, its limit r must satisfy the equation $r = 1/(20\sqrt{3}/3 - 12r)$, i.e., either $r = 1/6\sqrt{3}$ or $r = \sqrt{3}/2$. But, from (20), it is deduced that the limit of $\{\lambda_n/\sqrt{n}\}$, if it exists, is $1/6\sqrt{3}$. Then, to conclude the proof of this Proposition it only remains to prove that $\{\lambda_n/\sqrt{n}\}$ converges.

If $r = 1/6\sqrt{3}$ and $\theta = \frac{\sqrt{5}}{3}$, then, from (24), we have

$$\left|\frac{\lambda_n}{\sqrt{n}} - r\right| = \frac{|1 - rB(n) + rC(n)\frac{\lambda_{n-2}}{\sqrt{n-2}} - r^2 C(n) + r^2 C(n)|}{B(n) - C(n)\frac{\lambda_{n-2}}{\sqrt{n-2}}}. \tag{27}$$

On the other hand, using inequality (20) for the sequence $\frac{\lambda_n}{\sqrt{n}}$, we have $C(n)\frac{\lambda_{n-2}}{\sqrt{n-2}} < \theta C(n)$ and so, for n large enough,

$$B(n) - C(n)\frac{\lambda_{n-2}}{\sqrt{n-2}} > B(n) - \theta C(n) > 0.$$

From here we can give a bound for (27), i.e., for n large enough,

$$\left|\frac{\lambda_n}{\sqrt{n}} - r\right| < \frac{|1 - rB(n) + r\,C(n)(\frac{\lambda_{n-2}}{\sqrt{n-2}} - r) + r^2 C(n)|}{B(n) - \theta C(n)}$$
$$\leqslant \frac{|1 - rB(n) + r^2 C(n)|}{B(n) - \theta C(n)} + \frac{rC(n)}{B(n)-\theta C(n)}\left|\frac{\lambda_{n-2}}{\sqrt{n-2}} - r\right|.$$

Now, taking into account the limits of the sequences $B(n)$ and $C(n)$ given in (26), we obtain

$$\limsup_{n\to\infty}\left|\frac{\lambda_n}{\sqrt{n}} - r\right| \leqslant \frac{2/\sqrt{3}}{\frac{20}{\sqrt{3}} - 4\sqrt{5}} \limsup_{n\to\infty}\left|\frac{\lambda_{n-2}}{\sqrt{n-2}} - r\right|$$
$$= \frac{1}{10 - 2\sqrt{15}} \limsup_{n\to\infty}\left|\frac{\lambda_{n-2}}{\sqrt{n-2}} - r\right|,$$

where

$$\frac{1}{10 - 2\sqrt{15}} < \frac{1}{2},$$

and, therefore, we can conclude that the sequence $\{\lambda_n/\sqrt{n}\}$ is convergent and its limit is $r = 1/(6\sqrt{3})$. □

We want to compare the asymptotic behavior of Q_n and P_n in the complex plane, more exactly, in $\mathbb{C}\backslash\mathbb{R}$. We get the following result:

Theorem 2. *The asymptotic behavior*

$$\lim_{n\to\infty} \frac{Q_n(x)}{P_n(x)} = \frac{3}{2} \tag{28}$$

holds uniformly on compact subsets of $\mathbb{C}\backslash\mathbb{R}$.

Proof. We consider the orthonormal polynomials p_n with respect to the inner product $\int_{-\infty}^{\infty} f(x)g(x)e^{-x^4}\,dx$. In [5] López and Rakhmanov give the strong asymptotics of p_n, i.e., it holds uniformly on compact subsets of $\mathbb{C}\backslash\mathbb{R}$,

$$\lim_{n\to\infty} \frac{p_n(x)}{D_n(x)(\varphi(x/x_n))^{n+1/2}} = \frac{1}{\sqrt{2\pi}},$$

where $D_n(x)$ is the Szegő's function for the weight e^{-x^4} on the segment $[-x_n, x_n]$, i.e.,

$$D_n(x) = \exp\left\{\frac{\sqrt{x^2 - x_n^2}}{2\pi}\int_{-x_n}^{x_n} \frac{t^4}{(x-t)\sqrt{x_n^2 - t^2}}\,dt\right\},$$

being $x_n = (\frac{4n}{3})^{1/4}$ and $\varphi(x) = x + \sqrt{x^2 - 1}$ is the conformal mapping of $\mathbb{C}\backslash[-1, 1]$ onto the exterior of the closed unit disk.

Thus, we can deduce that

$$\lim_{n\to\infty} \frac{p_n(x)}{p_{n+2}(x)} = -1,$$

uniformly on compact subsets of $\mathbb{C}\setminus\mathbb{R}$. Then, for the monic polynomials P_n we get

$$\lim_{n\to\infty} \frac{\sqrt{n}P_n(x)}{P_{n+2}(x)} = -2\sqrt{3}, \tag{29}$$

uniformly on compact subsets of $\mathbb{C}\setminus\mathbb{R}$. Dividing relation (11) by $P_n(x)$ we obtain

$$\frac{Q_n(x)}{P_n(x)} = 1 - \lambda_{n-2}\frac{P_{n-2}(x)}{P_n(x)}\frac{Q_{n-2}(x)}{P_{n-2}(x)},$$

where, using (16) and (29), we have

$$\lim_{n\to\infty} \lambda_{n-2}\frac{P_{n-2}(x)}{P_n(x)} = -\frac{1}{3},$$

uniformly on compact subsets of $\mathbb{C}\setminus\mathbb{R}$. Now, standard arguments allow us to conclude that the sequence $\{Q_n/P_n\}$ is convergent and its limit is the solution of the equation $s = 1 + s/3$, that is, $s = 3/2$. □

From this theorem we deduce that the Sobolev polynomials $\{Q_n\}$ have the same asymptotic behavior (up to multiplicative constant factors) as $\{P_n\}$ in $\mathbb{C}\setminus\mathbb{R}$. This occurs in other cases when the measures μ_0 and μ_1 involved in the Sobolev inner product (1) with $N = 1$ have unbounded support (see [1,6]) but this is not the case when the measures have compact support (see, for example, [11] or [12]). Three natural questions arise. The first one is why does it occur? The second one is when does it occur? Finally, can we give a more complete description of the asymptotic behavior of the polynomials Q_n? The answer to the first and second questions is yet open for us, but we can obtain better information about the asymptotics of Q_n if we scale the variable x in a convenient way, i.e., if we look for the exterior Plancherel–Rotach-type asymptotics for Q_n. We have the following result:

Theorem 3. *The asymptotic behavior*

$$\lim_{n\to\infty} \frac{Q_n(\sqrt[4]{n}x)}{P_n(\sqrt[4]{n}x)} = \frac{3\varphi^2\left(\sqrt[4]{\frac{3}{4}}\,x\right)}{3\varphi^2\left(\sqrt[4]{\frac{3}{4}}\,x\right)+1} \tag{30}$$

holds uniformly on compact subsets of $\mathbb{C}\setminus[-\sqrt[4]{4/3}, \sqrt[4]{4/3}]$, *where* $\varphi(x) = x + \sqrt{x^2-1}$ *with* $\sqrt{x^2-1} > 0$ *if* $x > 1$, *i.e., the conformal mapping of* $\mathbb{C}\setminus[-1,1]$ *onto the exterior of the closed unit disk.*

Proof. It is well-known (see [16]) that from the three-term recurrence relation

$$xp_n(x) = a_{n+1}p_{n+1}(x) + a_np_{n-1}(x), \quad n\geqslant 1,$$

we can obtain asymptotic properties of the orthonormal polynomials p_n. Indeed, as

$$\lim_{n\to\infty} \frac{a_n}{\sqrt[4]{n+j}} = \frac{1}{\sqrt[4]{12}}, \quad \text{for a } j\in\mathbb{R} \text{ fixed},$$

we deduce (see [16])

$$\lim_{n\to\infty} \frac{p_{n-1}(\sqrt[4]{n+j}x)}{p_n(\sqrt[4]{n+j}x)} = \frac{1}{\varphi\left(\sqrt[4]{\frac{3}{4}}x\right)}, \quad j \text{ fixed},$$

uniformly on compact subsets of $\mathbb{C}\backslash[-\sqrt[4]{4/3}, \sqrt[4]{4/3}]$. Then, for the monic polynomial P_n we have

$$\lim_{n\to\infty} \sqrt[4]{n}\frac{P_{n-1}(\sqrt[4]{n+j}\,x)}{P_n(\sqrt[4]{n+j}\,x)} = \frac{\sqrt[4]{12}}{\varphi\left(\sqrt[4]{\frac{3}{4}}x\right)}, \quad j \text{ fixed}, \tag{31}$$

uniformly on compact subsets of $\mathbb{C}\backslash[-\sqrt[4]{4/3}, \sqrt[4]{4/3}]$. Introducing the change of variable $x\to\sqrt[4]{n}x$ in (11) and using this relation in a recursive way, we get

$$Q_n(\sqrt[4]{n}\,x) = \sum_{j=0}^{[(n-1)/2]} (-1)^j b_{2j}^{(n)} P_{n-2j}(\sqrt[4]{n}\,x), \quad n\geqslant 3,$$

with $b_0^{(n)} = 1$, $b_{2j}^{(n)} = \prod_{i=1}^{j} \lambda_{n-2i}$, and $[a]$ denotes the integer part of a. Then, dividing by $P_n(\sqrt[4]{n}\,x)$ we obtain

$$\begin{aligned}\frac{Q_n(\sqrt[4]{n}\,x)}{P_n(\sqrt[4]{n}\,x)} &= \sum_{j=0}^{[(n-1)/2]} (-1)^j b_{2j}^{(n)} \frac{P_{n-2j}(\sqrt[4]{n}\,x)}{P_n(\sqrt[4]{n}\,x)} \\ &= \sum_{j=0}^{[(n-1)/2]} (-1)^j \frac{b_{2j}^{(n)}}{\prod_{i=1}^{j} \sqrt{n-2i}} \frac{\prod_{i=1}^{j} \sqrt{n-2i}\, P_{n-2j}(\sqrt[4]{n}\,x)}{P_n(\sqrt[4]{n}\,x)},\end{aligned}$$

where an empty product is equal to 1. Now, we analyze the asymptotic behavior of the factors in the above sum. If we use (16) in Proposition 3 and (31), then

$$\lim_{n\to\infty} (-1)^j \frac{b_{2j}^{(n)}}{\prod_{i=1}^{j} \sqrt{n-2i}} = (-1)^j \prod_{i=1}^{j} \frac{\lambda_{n-2i}}{\sqrt{n-2i}} = \left(\frac{-1}{6\sqrt{3}}\right)^j, \quad j \text{ fixed},$$

$$\lim_{n\to\infty} \prod_{i=1}^{j} \sqrt{n-2i}\frac{P_{n-2j}(\sqrt[4]{n}\,x)}{P_n(\sqrt[4]{n}\,x)} = \left(\frac{2\sqrt{3}}{\varphi^2\left(\sqrt[4]{\frac{3}{4}}x\right)}\right)^j, \quad j \text{ fixed}. \tag{32}$$

This last limit holds uniformly on compact subsets of $\mathbb{C}\setminus[-\sqrt[4]{4/3}, \sqrt[4]{4/3}]$. Gathering the above limits we get

$$\lim_{n\to\infty} (-1)^j b_{2j}^{(n)} \frac{P_{n-2j}(\sqrt[4]{n}\,x)}{P_n(\sqrt[4]{n}\,x)} = \left(\frac{-1}{3\,\varphi^2\left(\sqrt[4]{\frac{3}{4}}\,x\right)}\right)^j, \tag{33}$$

uniformly on compact subsets of $\mathbb{C}\setminus[-\sqrt[4]{4/3}, \sqrt[4]{4/3}]$.

On the other hand, the upper bound of the sequence $\{\lambda_n/\sqrt{n}\}$ obtained in Proposition 3 together with the limit relation (32) allow us to give a uniform bound for $(-1)^j b_{2j}^{(n)} P_{n-2j}(\sqrt[4]{n}x)/P_n(\sqrt[4]{n}x)$ on $\mathbb{C}\setminus[-\sqrt[4]{4/3}, \sqrt[4]{4/3}]$, that is, for n large enough and $0\leqslant j\leqslant[(n-1)/2]$,

$$\left|(-1)^j b_{2j}^{(n)} \frac{P_{n-2j}(\sqrt[4]{n}\,x)}{P_n(\sqrt[4]{n}\,x)}\right| \leqslant K\theta^j,$$

where

$$\theta = \frac{\sqrt{5}}{3} < 1$$

and K is a constant. Therefore, we have a majorant for $Q_n(\sqrt[4]{n}\,x)/P_n(\sqrt[4]{n}\,x)$ with $x\in\mathbb{C}\setminus[-\sqrt[4]{4/3}, \sqrt[4]{4/3}]$. From Lebesgue's dominated convergence theorem together with (33) we get

$$\lim_{n\to\infty} \frac{Q_n(\sqrt[4]{n}\,x)}{P_n(\sqrt[4]{n}\,x)} = \sum_{j=0}^{\infty}\left(\frac{-1}{3\varphi^2\left(\sqrt[4]{\frac{3}{4}}\,x\right)}\right)^j = \frac{3\varphi^2\left(\sqrt[4]{\frac{3}{4}}\,x\right)}{3\varphi^2\left(\sqrt[4]{\frac{3}{4}}\,x\right)+1},$$

uniformly on compact subsets of $\mathbb{C}\setminus[-\sqrt[4]{4/3}, \sqrt[4]{4/3}]$ and thus the statement of our theorem follows. Note that $\left|-1/\left(3\varphi^2\left(\sqrt[4]{\frac{3}{4}}\,x\right)\right)\right| < 1$ when $x\in\mathbb{C}\setminus[-\sqrt[4]{4/3}, \sqrt[4]{4/3}]$. □

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